

THE FIRST HOMOLOGY GROUP OF THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE WITH TWISTED COEFFICIENTS

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ABSTRACT. We determine the first homology group of the mapping class group $\mathcal{M}(N)$ of a nonorientable surface N with coefficients in $H_1(N; \mathbb{Z})$.

1. INTRODUCTION

Let $N_{g,s}$ be a smooth, nonorientable, compact surface of genus g with s boundary components. If s is zero, then we omit it from the notation. If we do not want to emphasise the numbers g, s , we simply write N for a surface $N_{g,s}$. Recall that N_g is a connected sum of g projective planes and $N_{g,s}$ is obtained from N_g by removing s open discs.

Let $\text{Diff}(N)$ be the group of all diffeomorphisms $h: N \rightarrow N$ such that h is the identity on each boundary component. By $\mathcal{M}(N)$ we denote the quotient group of $\text{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies are the identity on each boundary component. $\mathcal{M}(N)$ is called the *mapping class group* of N .

The mapping class group $\mathcal{M}(S_{g,s})$ of an orientable surface is defined analogously, but we consider only orientation preserving maps.

1.1. Background. Homological computations play a prominent role in the theory of mapping class groups. In the orientable case, Mumford [10] observed that $H_1(\mathcal{M}(S_g))$ is a quotient of \mathbb{Z}_{10} . Then Birman [1, 2] showed that if $g \geq 3$ then $H_1(\mathcal{M}(S_g))$ is a quotient of \mathbb{Z}_2 , and Powell [12] showed that in fact $H_1(\mathcal{M}(S_g))$ is trivial if $g \geq 3$. As for higher homology groups, Harer [4, 5] computed $H_i(\mathcal{M}(S_g))$ for $i = 2, 3$ and Madsen and Weiss [8] determined the integral cohomology ring of the stable mapping class group.

In the nonorientable case, Korkmaz [6, 7] computed $H_1(\mathcal{M}(N_g))$ for a closed surface N_g (possibly with marked points). This computation was later [13] extended to the case of a surface with boundary. As for higher homology groups, Wahl [17] identified the stable rational cohomology of $\mathcal{M}(N)$.

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As for twisted coefficients, Morita [9] proved that

$$H_1(\mathcal{M}(S_g); H_1(S_g; \mathbb{Z})) \cong \mathbb{Z}_{2g-2}, \quad \text{for } g \geq 2.$$

There are also similar computations for the hyperelliptic mapping class groups. Tanaka [16] showed that $H_1(\mathcal{M}^h(S_g); H_1(S_g; \mathbb{Z})) \cong \mathbb{Z}_2$ for $g \geq 2$ and in the nonorientable case we showed in [15] that

$$H_1(\mathcal{M}^h(N_g); H_1(N_g; \mathbb{Z})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \text{for } g \geq 3.$$

1.2. Main results. The purpose of this paper is to prove the following theorem.

Theorem 1.1. *If $N_{g,s}$ is a nonorientable surface of genus $g \geq 3$ with $s \leq 1$ boundary components, then*

$$H_1(\mathcal{M}(N_{g,s}); H_1(N_{g,s}; \mathbb{Z})) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \in \{3, 4, 5, 6\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \geq 7. \end{cases}$$

The starting point for this computation is the presentation for the mapping class group $\mathcal{M}(N_{g,s})$, where $g + s \geq 3$ and $s \in \{0, 1\}$, obtained recently by Paris and Szepietowski [11] (Theorems 2.1 and 2.2).

2. PRELIMINARIES

2.1. Nonorientable surfaces. Represent surfaces $N_{g,0}$ and $N_{g,1}$ as respectively a sphere or a disc with g crosscaps and let $\alpha_1, \dots, \alpha_{g-1}, \beta_1, \dots, \beta_{\frac{g-2}{2}}$ be two-sided circles indicated in Figures 1 and 2. Small arrows in these fig-

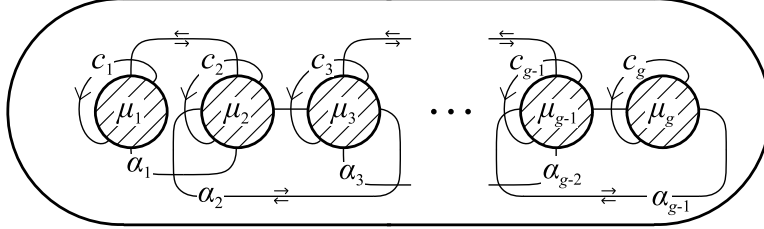


FIGURE 1. Surface N_g [$N_{g,1}$] as a sphere [disc] with crosscaps.

ures indicate directions of Dehn twists $a_1, \dots, a_{g-1}, b_1, \dots, b_{\frac{g-2}{2}}$ associated with these circles.

For any two consecutive crosscaps μ_i, μ_{i+1} we define a *crosscap transposition* u_i to be the map which interchanges these two crosscaps (see Figure 3). Moreover, if N_g is closed, then we define

$$\varrho = a_1 a_2 \cdots a_{g-1} u_{g-1} \cdots u_2 u_1.$$

Element ϱ represents the *hyperelliptic involution*, that is the reflection of N_g across the plane containing the centers of crosscaps in Figure 1 – for details see [15].

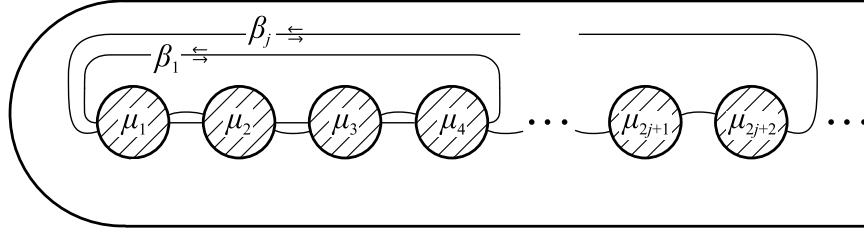
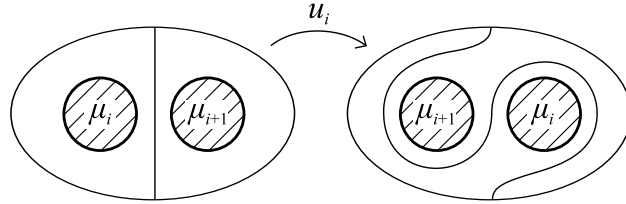
FIGURE 2. Circles $\beta_1, \beta_2, \dots, \beta_{\frac{g-2}{2}}$.

FIGURE 3. Crosscap transposition.

Recently Paris and Szepietowski [11] obtained a simple presentation for the mapping class group $\mathcal{M}(N_{g,s})$ if $g + s > 3$ and $s \in \{0, 1\}$. The next two theorems state a slightly simplified version of Paris-Szepietowski presentation (see Proposition 3.3 and Corollaries 3.2, 3.4 of [14]).

Theorem 2.1. *If $g \geq 3$ is odd or $g = 4$, then $\mathcal{M}(N_{g,1})$ admits a presentation with generators: $a_1, \dots, a_{g-1}, u_1, \dots, u_{\min\{5, g-1\}}$ and b_1 for $g \geq 4$. The defining relations are:*

- (A1) $a_j a_k = a_k a_j$ for $g \geq 4$, $|j - k| > 1$,
- (A2) $a_j a_{j+1} a_j = a_{j+1} a_j a_{j+1}$ for $j = 1, \dots, g - 2$,
- (A3) $a_j b_1 = b_1 a_j$ for $g \geq 4$, $j \neq 4$,
- (A4) $b_1 a_4 b_1 = a_4 b_1 a_4$ for $g \geq 5$,
- (A5) $(a_2 a_3 a_4 b_1)^{10} = (a_1 a_2 a_3 a_4 b_1)^6$ for $g \geq 5$,
- (A6) $(a_2 a_3 a_4 a_5 a_6 b_1)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b_1)^9$ for $g \geq 7$,
- (B1) $u_1 u_3 = u_3 u_1$ for $g \geq 4$,
- (B2) $u_1 u_2 u_1 = u_2 u_1 u_2$,
- (C1) $u_1 a_j = a_j u_1$ for $g \geq 4$, $j = 3, \dots, g - 1$,
- (C2) $a_1 u_2 u_1 = u_2 u_1 a_2$,
- (C4) $a_1 u_1 a_1 = u_1$,
- (C5) $u_{j+1} a_j a_{j+1} u_j = a_j a_{j+1}$ for $j = 1, \dots, \min\{5, g - 1\}$,
- (C6) $(u_3 b_1)^2 = (a_1 a_2 a_3)^2 (u_1 u_2 u_3)^2$ for $g \geq 4$,
- (C7) $u_5 b_1 = b_1 u_5$ for $g \geq 6$,
- (C8) $b_1 a_4 u_4 = a_4 u_4 (a_4 a_3 a_2 a_1 u_1 u_2 u_3 u_4) b_1$ for $g \geq 5$.

If $g \geq 6$ is even, then $\mathcal{M}(N_{g,1})$ admits a presentation with generators: $a_1, \dots, a_{g-1}, u_1, \dots, u_5$ and additionally: $b_0, b_1, \dots, b_{\frac{g-2}{2}}$. The defining relations are relations: (A1)–(A6), (B1)–(B2), (C1)–(C8) above and additionally:

$$(A7) \quad b_0 = a_1,$$

$$(A8) \quad b_{j+1}(b_{j-1}a_{2j}a_{2j+1}a_{2j+2}a_{2j+3})^6 = (b_{j-1}a_{2j}a_{2j+1}a_{2j+2}a_{2j+3}b_j)^5 \\ \text{for } 1 \leq j \leq \frac{g-4}{2},$$

$$(A9a) \quad b_2b_1 = b_1b_2 \quad \text{for } g = 6,$$

$$(A9b) \quad b_{\frac{g-2}{2}}a_{g-5} = a_{g-5}b_{\frac{g-2}{2}} \quad \text{for } g \geq 8.$$

Theorem 2.2. *If $g \geq 4$, then the group $\mathcal{M}(N_{g,0})$ is isomorphic to the quotient of the group $\mathcal{M}(N_{g,1})$ with the presentation given in Theorem 2.1 obtained by adding a generator ϱ and relations:*

$$(B3) \quad (a_1a_2 \cdots a_{g-1})^g = \begin{cases} 1 & \text{for } g \text{ even} \\ \varrho & \text{for } g \text{ odd,} \end{cases}$$

$$(D1) \quad \varrho a_j = a_j \varrho \quad \text{for } j = 1, \dots, g-1,$$

$$(D2) \quad u_1 \varrho u_1 = \varrho,$$

$$(E) \quad \varrho^2 = 1,$$

$$(F) \quad (u_1a_1a_2a_3 \cdots a_{g-1}\varrho)^{g-1} = 1.$$

2.2. Homology of groups. Let us briefly review how to compute the first homology of a group with twisted coefficients – for more details see Section 5 of [15] and references there.

For a given group G and G -module M (that is $\mathbb{Z}G$ -module) we define $C_2(G), C_1(G)$ as the free G -modules generated respectively by symbols $[h_1|h_2]$ and $[h_1]$, where $h_i \in G$. We define also $C_0(G)$ as the free G -module generated by the empty bracket $[\cdot]$. Then the first homology group $H_1(G; M)$ is the first homology group of the complex

$$C_2(G) \otimes_G M \xrightarrow{\partial_2 \otimes \text{id}} C_1(G) \otimes_G M \xrightarrow{\partial_1 \otimes \text{id}} C_0(G) \otimes_G M,$$

where

$$\partial_2([h_1|h_2]) = h_1[h_2] - [h_1h_2] + [h_1],$$

$$\partial_1([h]) = h[\cdot] - [\cdot].$$

For simplicity, we write $\otimes_G = \otimes$ and $\partial \otimes \text{id} = \bar{\partial}$ henceforth.

If the group G has a presentation $G = \langle X | R \rangle$ and

$$\langle \bar{X} \rangle = \langle [x] \otimes m \mid x \in X, m \in M \rangle \subseteq C_1(G) \otimes M,$$

then $H_1(G; M)$ is a quotient of $\langle \bar{X} \rangle \cap \ker \bar{\partial}_1$.

The kernel of this quotient corresponds to relations in G (that is elements of R). To be more precise, if $r \in R$ has the form $x_1 \cdots x_k = y_1 \cdots y_n$ and $m \in M$, then r gives the relation (in $H_1(G; M)$)

$$(2.1) \quad \bar{r} \otimes m: \sum_{i=1}^k x_1 \cdots x_{i-1} [x_i] \otimes m = \sum_{i=1}^n y_1 \cdots y_{i-1} [y_i] \otimes m.$$

Then

$$H_1(G; M) = \langle \bar{X} \rangle \cap \ker \bar{\partial}_1 / \langle \bar{R} \rangle,$$

where

$$\bar{R} = \{ \bar{r} \otimes m \mid r \in R, m \in M \}.$$

3. ACTION OF $\mathcal{M}(N_{g,s})$ ON $H_1(N_{g,s}; \mathbb{Z})$

Let c_1, \dots, c_g be one-sided circles indicated in Figure 1. Recall that if $s \in \{0, 1\}$, then the \mathbb{Z} -module $H_1(N_{g,s}; \mathbb{Z})$ is generated by $\gamma_1 = [c_1], \dots, \gamma_g = [c_g]$. If $s = 1$ then $\gamma_1, \dots, \gamma_g$ are free generators of $H_1(N_{g,s}; \mathbb{Z})$, and if $s = 0$ then they generate $H_1(N_{g,s}; \mathbb{Z})$ with respect to the single relation

$$2(\gamma_1 + \gamma_2 + \dots + \gamma_g) = 0.$$

The mapping class group $\mathcal{M}(N_{g,s})$ acts on $H_1(N_{g,s}; \mathbb{Z})$, hence we have a representation

$$\psi: \mathcal{M}(N_{g,s}) \rightarrow \text{Aut}(H_1(N_{g,s}; \mathbb{Z})).$$

It is straightforward to check that

$$\begin{aligned}
 \psi(a_j) &= I_{j-1} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \oplus I_{g-j-1} \\
 \psi(a_j^{-1}) &= I_{j-1} \oplus \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \oplus I_{g-j-1} \\
 \psi(u_j) &= \psi(u_j^{-1}) = I_{j-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{g-j-1} \\
 \psi(b_j) &= I_g + \begin{bmatrix} -1 & 1 & -1 & \dots & 1 \\ -1 & 1 & -1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & -1 & \dots & 1 \end{bmatrix}_{2j+2} \oplus 0_{g-(2j+2)} \\
 \psi(b_j^{-1}) &= I_g + \begin{bmatrix} 1 & -1 & 1 & \dots & -1 \\ 1 & -1 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & 1 & \dots & -1 \end{bmatrix}_{2j+2} \oplus 0_{g-(2j+2)} \\
 \psi(\varrho) &= \psi(\varrho^{-1}) = -I_g
 \end{aligned} \tag{3.1}$$

where I_k is the identity matrix of rank k .

4. COMPUTING $\langle \bar{X} \rangle \cap \ker \bar{\partial}_1$

If $h \in G$, then

$$\bar{\partial}_1([h] \otimes \gamma_i) = (h - 1)[\cdot] \otimes \gamma_i = (\psi(h)^{-1} - I_g)\gamma_i,$$

where we identified $C_0(G) \otimes M$ with M by the map $[\cdot] \otimes m \mapsto m$.

Let us denote $[a_j] \otimes \gamma_i, [u_j] \otimes \gamma_i, [\varrho] \otimes \gamma_i$ respectively by $a_{j,i}, u_{j,i}, \varrho_i$. Using formulas 3.1, we obtain

$$(4.1) \quad \begin{aligned} \bar{\partial}_1(a_{j,i}) &= \begin{cases} \gamma_j + \gamma_{j+1} & \text{if } i = j \\ -\gamma_j - \gamma_{j+1} & \text{if } i = j + 1 \\ 0 & \text{otherwise,} \end{cases} \\ \bar{\partial}_1(u_{j,i}) &= \begin{cases} -\gamma_j + \gamma_{j+1} & \text{if } i = j \\ \gamma_j - \gamma_{j+1} & \text{if } i = j + 1 \\ 0 & \text{otherwise,} \end{cases} \\ \bar{\partial}_1(b_{j,i}) &= \begin{cases} \gamma_1 + \gamma_2 + \cdots + \gamma_{2j+2} & \text{if } i = 1, 3, \dots, 2j + 1 \\ -\gamma_1 - \gamma_2 + \cdots - \gamma_{2j+2} & \text{if } i = 2, 4, \dots, 2j + 2 \\ 0 & \text{otherwise,} \end{cases} \\ \bar{\partial}_1(\varrho_i) &= -2\gamma_i. \end{aligned}$$

Proposition 4.1. *Let $g \geq 4$ and $G = \mathcal{M}(N_{g,0})$. Then $\langle \bar{X} \rangle \cap \ker \bar{\partial}_1$ is the abelian group which admits the presentation with generators:*

- (G1) $a_{j,i}$ for $j = 1, \dots, g-1$ and $i = 1, \dots, j-1, j+2, \dots, g$,
- (G2) $a_{j,j} + a_{j,j+1}$ for $j = 1, \dots, g-1$,
- (G3) $u_{j,i}$ for $j = 1, 2, \dots, \min\{5, g-1\}$ and $i = 1, \dots, j-1, j+2, \dots, g$,
- (G4) $u_{j,j} + u_{j,j+1}$ for $j = 1, 2, \dots, \min\{5, g-1\}$,
- (G5) $a_{j,j} - a_{j+1,j+1} + u_{j,j} + u_{j+1,j+1}$ for $j = 1, 2, \dots, \min\{5, g-1\}$,
- (G6) $2a_{j,j} + \varrho_j + \varrho_{j+1}$ for $j = 1, \dots, g-1$,
- (G7) $a_{1,1} + \varrho_1 = u_{1,1}$,
- (G8) $b_{j,i}$ for $i = 2j+3, \dots, g$,
- (G9) $b_{j,2i} + b_{j,1}$ for $i = 1, \dots, j+1$,
- (G10) $b_{j,2i+1} - b_{j,1}$ for $i = 1, \dots, j$,
- (G11) $b_{j,1} - a_{1,1} - a_{3,3} - \cdots - a_{2j+1,2j+1}$,
- (G12) $\begin{cases} a_{1,1} + 2a_{2,2} + 2a_{4,4} + \cdots + 2a_{g-1,g-1} - u_{1,1} & \text{if } g \text{ is odd} \\ 2a_{1,1} + 2a_{3,3} + \cdots + 2a_{g-1,g-1} & \text{if } g \text{ is even,} \end{cases}$

and relations:

$$r_{a_j} : 0 = 2a_{j,1} + \cdots + 2(a_{j,j} + a_{j,j+1}) + \cdots + 2a_{j,g} \quad \text{for } j = 1, \dots, g-1$$

$$r_{u_j} : 0 = 2u_{j,1} + \cdots + 2(u_{j,j} + u_{j,j+1}) + \cdots + 2u_{j,g} \\ \text{for } j = 1, \dots, \min\{5, g-1\}$$

$$r_{b_j} : 0 = 2(b_{j,2} + b_{j,1}) + \cdots + 2(b_{j,2j+2} + b_{j,1}) \\ + 2(b_{j,3} - b_{j,1}) + \cdots + 2(b_{j,2j+1} - b_{j,1}) + 2b_{j,2j+3} + \cdots + 2b_{j,g}$$

$$r_{\varrho} : \begin{cases} 2(a_{1,1} + \varrho_1 - u_{1,1}) + 2(2a_{2,2} + \varrho_2 + \varrho_3) + \cdots + 2(2a_{g-1,g-1} + \varrho_{g-2} + \varrho_g) \\ \quad = 2(a_{1,1} + 2a_{2,2} + \cdots + 2a_{g-1,g-1} - u_{1,1}) & \text{if } g \text{ is odd} \\ 2(2a_{1,1} + \varrho_1 + \varrho_2) + \cdots + 2(2a_{g-1,g-1} + \varrho_{g-1} + \varrho_g) \\ \quad = 2(2a_{1,1} + 2a_{3,3} + \cdots + 2a_{g-1,g-1}) & \text{if } g \text{ is even.} \end{cases}$$

Proof. By Theorem 2.2, $\langle \overline{X} \rangle$ is generated by $a_{j,i}, u_{j,i}, b_{j,i}$ and ϱ_i . Using formulas 4.1, it is straightforward to check that elements (G1)–(G12) are elements of $\ker \overline{\partial}_1$. Moreover,

$$2a_{j,1} + 2a_{j,2} + \cdots + 2a_{j,g} = [a_j] \otimes 2(\gamma_1 + \cdots + \gamma_g) = 0,$$

hence r_{a_j} is indeed a relation. Similarly we check that r_{u_j}, r_{b_j} and r_{ϱ} are relations.

Observe that using relations $r_{a_j}, r_{u_j}, r_{b_j}$ and r_{ϱ} we can substitute for $2a_{j,g}, 2u_{j,g}, 2b_{j,g}$ and $2\varrho_g$ respectively, hence each element in $\langle \overline{X} \rangle$ can be written as a linear combination of $a_{j,i}, u_{j,i}, b_{j,i}, \varrho_i$, where each of $a_{j,g}, u_{j,g}, b_{j,g}, \varrho_g$ has the coefficient 0 or 1. Moreover, for a given $x \in \langle \overline{X} \rangle \subset C_1(G) \otimes H_1(N_g; \mathbb{Z})$ such a combination is unique. Hence for the rest of the proof we assume that linear combinations of $a_{j,i}, u_{j,i}, b_{j,i}, \varrho_j$ satisfy this condition.

Suppose that $h \in \langle \overline{X} \rangle \cap \ker \overline{\partial}_1$. We will show that h can be uniquely expressed as a linear combination of generators (G1)–(G12).

We decompose h as follows:

- $h = h_0 = h_1 + h_2$, where h_1 is a combination of generators (G1)–(G2) and h_2 does not contain $a_{j,i}$ with $i \neq j$;
- $h_2 = h_3 + h_4$, where h_3 is a combination of generators (G3)–(G4) and h_4 does not contain $u_{j,i}$ with $i \neq j$;
- $h_4 = h_5 + h_6$, where h_5 is a combination of generators (G5) and h_6 does not contain $u_{j,j}$ for $j > 1$;
- $h_6 = h_7 + h_8$, where h_7 is a combination of generators (G6) and h_8 does not contain ϱ_j for $j > 1$;
- $h_8 = h_9 + h_{10}$, where h_9 is a multiple of generator (G7) and h_{10} does not contain ϱ_1 ;
- $h_{10} = h_{11} + h_{12}$, where h_{11} is a combination of generators (G8)–(G10) and h_{12} does not contain $b_{j,i}$ for $i \neq 1$;
- $h_{12} = h_{13} + h_{14}$, where h_{13} is a combination of generators (G11) and h_{14} does not contain $b_{j,i}$.

Observe also that for each $k = 0, \dots, 12$, h_{k+1} and h_{k+2} are uniquely determined by h_k . Element h_{14} has the form

$$h_{14} = \sum_{j=1}^{g-1} \alpha_j a_{j,j} + \alpha u_{1,1}$$

for some integers $\alpha, \alpha_1, \dots, \alpha_{g-1}$. Hence

$$0 = \overline{\partial}_1(h_{14}) = (\alpha_1 - \alpha)\gamma_1 + (\alpha_1 + \alpha_2 + \alpha)\gamma_2 + (\alpha_2 + \alpha_3)\gamma_3 + \cdots + \alpha_{g-1}\gamma_g.$$

If g is odd this implies that

$$\alpha_2 = \alpha_4 = \cdots = \alpha_{g-1} = 2k, \quad \alpha_3 = \alpha_5 = \cdots = \alpha_{g-2} = 0, \quad \alpha_1 = k, \quad \alpha = -k$$

for some $k \in \mathbb{Z}$. For g even we get

$$\alpha_1 = \alpha_3 = \alpha_5 = \cdots = \alpha_{g-1} = 2k, \quad \alpha_2 = \alpha_4 = \cdots = \alpha_{g-2} = 0, \quad \alpha = 0.$$

In both cases h_{14} is a multiple of the generator (G12). \square

By an analogous argument we get

Proposition 4.2. *Let $g \geq 3$ and $G = \mathcal{M}(N_{g,1})$. Then $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$ is the abelian group generated freely by:*

- (G1) $a_{j,i}$ for $j = 1, \dots, g-1$ and $i = 1, \dots, j-1, j+2, \dots, g$,
- (G2) $a_{j,j} + a_{j,j+1}$ for $j = 1, \dots, g-1$,
- (G3) $u_{j,i}$ for $j = 1, 2, \dots, \min\{5, g-1\}$ and $i = 1, \dots, j-1, j+2, \dots, g$,
- (G4) $u_{j,j} + u_{j,j+1}$ for $j = 1, 2, \dots, \min\{5, g-1\}$,
- (G5) $a_{j,j} - a_{j+1,j+1} + u_{j,j} + u_{j+1,j+1}$ for $j = 1, 2, \dots, \min\{5, g-1\}$,
- (G8) $b_{j,i}$ for $i = 2j+3, \dots, g$,
- (G9) $b_{j,2i} + b_{j,1}$ for $i = 1, \dots, j+1$,
- (G10) $b_{j,2i+1} + b_{j,1}$ for $i = 1, \dots, j$,
- (G11) $b_{j,1} - a_{1,1} - a_{3,3} - \dots - a_{2j+1,2j+1}$.

5. COMPUTING $H_1(\mathcal{M}(N_{g,s}); H_1(N_{g,s}; \mathbb{Z}))$

Using formula (2.1) we rewrite relations from Theorems 2.1 and 2.2 as relations in $H_1(\mathcal{M}(N_{g,s}); H_1(N_{g,s}; \mathbb{Z}))$

(A1)–(A2). Relation (A1) gives

$$\begin{aligned}
 r_{j,k;i}^{(A1)} : 0 &= ([a_j] + a_j[a_k] - [a_k] - a_k[a_j]) \otimes \gamma_i \\
 &= a_{j,i} + [a_k] \otimes \psi(a_j^{-1})\gamma_i - a_{k,i} - [a_j] \otimes \psi(a_k^{-1})\gamma_i \\
 &= \pm \begin{cases} 0 & \text{if } i \notin \{j, j+1, k, k+1\} \\ a_{k,j} + a_{k,j+1} & \text{if } i \in \{j, j+1\} \\ a_{j,k} + a_{j,k+1} & \text{if } i \in \{k, k+1\}. \end{cases}
 \end{aligned}$$

Relation (A2) gives

$$\begin{aligned}
 r_{j,i}^{(A2)} : 0 &= ([a_j] + a_j[a_{j+1}] + a_j a_{j+1}[a_j] \\
 &\quad - [a_{j+1}] - a_{j+1}[a_j] - a_{j+1} a_j[a_{j+1}]) \otimes \gamma_i \\
 &= [a_j] \otimes (I_g + \psi(a_{j+1}^{-1} a_j^{-1}) - \psi(a_{j+1}^{-1}))\gamma_i \\
 &\quad + [a_{j+1}] \otimes (\psi(a_j^{-1}) - I_g - \psi(a_j^{-1} a_{j+1}^{-1}))\gamma_i \\
 &= \begin{cases} a_{j,i} - a_{j+1,i} & \text{if } i \notin \{j, j+1, j+2\} \\ a_{j,j+2} - a_{j+1,j} & \text{if } i = j+2 \\ (*) + 2(a_{j,j} + a_{j,j+1}) & \text{if } i = j \\ (*) - (a_{j,j} + a_{j,j+1}) - (a_{j+1,j+1} + a_{j+1,j+2}) & \text{if } i = j+1. \end{cases}
 \end{aligned}$$

In the above formula $(*)$ denotes some expression homologous to 0 by previously obtained relations. As we progress further, we will often perform simplifications based on previously obtained relations, from now on we will use symbol \equiv in such cases.

Carefully checking relations $r_{j,k;i}^{(A1)}$ and $r_{j,i}^{(A2)}$ we conclude that generators (G1) generate a cyclic group, and generators (G2) generate a cyclic group of order at most 2.

(C4). Relation (C4) gives

$$\begin{aligned}
 r_i^{(C4)} : 0 &= ([a_1] + a_1[u_1] + a_1u_1[a_1] - [u_1]) \otimes \gamma_i \\
 &= a_{1,i} + [u_1] \otimes \psi(a_1^{-1})\gamma_i + [a_1] \otimes \psi(u_1^{-1}a_1^{-1})\gamma_i - u_{1,i} \\
 &= \pm \begin{cases} (u_{1,1} + u_{1,2}) + 2(a_{1,1} + a_{1,2}) & \text{for } i = 1 \\ u_{1,1} + u_{1,2} & \text{for } i = 2 \\ 2a_{1,i} = 0 & \text{for } i > 2. \end{cases}
 \end{aligned}$$

Hence the cyclic group generated by generators (G1) has order at most 2 and generator (G4) is trivial for $j = 1$.

(C1). Relation (C1) gives

$$\begin{aligned}
 r_{j:i}^{(C1)} : 0 &= ([u_1] + u_1[a_j] - [a_j] - a_j[u_1]) \otimes \gamma_i \\
 &= u_{1,i} + [a_j] \otimes \psi(u_1^{-1})\gamma_i - a_{j,i} - [u_1] \otimes \psi(a_j^{-1})\gamma_i \\
 &= \pm \begin{cases} a_{j,1} - a_{j,2} & \text{for } i \in \{1, 2\} \\ u_{1,j} + u_{1,j+1} & \text{for } i \in \{j, j+1\} \\ 0 & \text{for } i \notin \{1, 2, j, j+1\}. \end{cases}
 \end{aligned}$$

The above relation implies that generators (G3) of the form $u_{1,j}$ where $j \geq 3$ generate a cyclic group (note that this statement holds also for $g = 3$).

(C5). Relation (C5) gives

$$\begin{aligned}
 r_{j:i}^{(C5)} : 0 &= ([u_{j+1}] + u_{j+1}[a_j] + u_{j+1}a_j[a_{j+1}] + u_{j+1}a_ja_{j+1}[u_j] - [a_j] - a_j[a_{j+1}]) \otimes \gamma_i \\
 &= u_{j+1,i} + [a_j] \otimes (\psi(u_{j+1}^{-1}) - I_g)\gamma_i + [a_{j+1}] \otimes (\psi(a_j^{-1}u_{j+1}^{-1}) - \psi(a_j^{-1}))\gamma_i \\
 &\quad + [u_j] \otimes \psi(a_{j+1}^{-1}a_j^{-1}u_{j+1}^{-1})\gamma_i \\
 &\equiv \begin{cases} u_{j+1,i} + u_{j,i} & \text{if } i \notin \{j, j+1, j+2\} \\ (u_{j+1,j} + u_{j,j+2}) + 2(u_{j,j} + u_{j,j+1}) & \text{if } i = j \\ (a_{j,j} - a_{j+1,j+1} + u_{j,j} + u_{j+1,j+1}) - (u_{j,j} + u_{j,j+1}) & \text{if } i = j+1 \\ -(a_{j,j} - a_{j+1,j+1} + u_{j,j} + u_{j+1,j+1}) + (u_{j+1,j+1} + u_{j+1,j+2}) & \text{if } i = j+2. \end{cases}
 \end{aligned}$$

From the last two equalities we obtain that generators (G4) generate a cyclic group, which by relation (C4) is trivial. Hence generators (G5) are also trivial.

Now we are going to show that generators (G3) generate a cyclic group.

If $k < i - 1$ then from the first equality and relation (C1) we have

$$u_{k,i} = -u_{k-1,i} = \cdots = \pm u_{1,i} = \pm u_{1,3}.$$

As for the elements $u_{k,i}$ with $k > i$, by the the first two equalities,

$$\begin{aligned} u_{5,4} &= -u_{4,6} \\ u_{5,3} &= -u_{4,3} = u_{3,5} \\ u_{5,2} &= -u_{4,2} = u_{3,2} = -u_{2,4} \\ u_{5,1} &= -u_{4,1} = u_{3,1} = -u_{2,1} = u_{1,3}. \end{aligned}$$

Hence in fact generators (G3) generate a cyclic group.

(B2). Relation (B2) gives

$$\begin{aligned} r_i^{(B2)} : 0 &= ([u_1] + u_1[u_2] + u_1u_2[u_1] - [u_2] - u_2[u_1] - u_2u_1[u_2]) \otimes \gamma_i \\ &= \begin{cases} u_{1,i} - u_{2,i} & \text{if } i > 3 \\ u_{1,3} - u_{2,1} & \text{if } i \in \{1, 3\} \\ (u_{1,1} + u_{1,2}) - (u_{2,2} + u_{2,3}) - (u_{1,3} - u_{2,1}) & \text{if } i = 2. \end{cases} \end{aligned}$$

In particular, $u_{1,3} = u_{2,1}$. But relation (C5) implies that $u_{1,3} = -u_{2,1}$. Hence the cyclic group generated by generators (G3) has order at most 2.

(B1). Relation (B1) gives

$$\begin{aligned} r_i^{(B1)} : 0 &= ([u_1] + u_1[u_3] - [u_3] - u_3[u_1]) \otimes \gamma_i \\ &= \pm \begin{cases} 0 & \text{if } i > 4 \\ u_{3,2} - u_{3,1} & \text{if } i \in \{1, 2\} \\ u_{1,4} - u_{1,3} & \text{if } i \in \{3, 4\}. \end{cases} \end{aligned}$$

This relation gives no new information.

(C2). Relation (C2) gives

$$\begin{aligned} r_i^{(C2)} : 0 &= ([a_1] + a_1[u_2] + a_1u_2[u_1] - [u_2] - u_2[u_1] - u_2u_1[a_2]) \otimes \gamma_i \\ &= \pm \begin{cases} a_{1,i} - a_{2,i} & \text{if } i > 3 \\ (a_{1,1} - a_{2,2} + u_{1,1} + u_{2,2}) + (u_{1,3} + u_{2,1}) & \text{if } i = 1 \\ (a_{1,1} + a_{1,2}) - (a_{2,2} + a_{2,3}) & \\ \quad - (a_{1,1} - a_{2,2} + u_{1,1} + u_{2,2}) - (u_{1,3} + u_{2,1}) & \text{if } i = 2 \\ a_{1,3} - a_{2,1} & \text{if } i = 3. \end{cases} \end{aligned}$$

This relation gives no new information.

Note that at this point we proved Theorem 1.1 for $N = N_{3,1}$,

$$H_1(\mathcal{M}(N_{3,1}); H_1(N_{3,1}; \mathbb{Z})) \cong \langle a_{1,3}, a_{1,1} + a_{1,2}, u_{1,3} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Observe also that for $N = N_{3,0}$, Theorem 1.1 follows from Theorem 5.4 of [15], hence from now we assume that $g \geq 4$.

(A3). Relation (A3) gives

$$\begin{aligned} r_{j,i}^{(A3)} : 0 &= ([a_j] + a_j[b_1] - [b_1] - b_1[a_j]) \otimes \gamma_i \\ &= a_{j,i} + [b_1] \otimes \psi(a_j^{-1})\gamma_i - b_{1,i} - [a_j] \otimes \psi(b_1^{-1})\gamma_i. \end{aligned}$$

If $i \neq j$ and $i \neq j + 1$, then we get

$$0 \equiv \begin{cases} (a_{1,1} + a_{1,2}) + (a_{1,3} + a_{1,4}) & \text{if } j = 1 \text{ and } i \in \{3, 4\} \\ (a_{2,2} + a_{2,3}) + (a_{2,1} + a_{2,4}) & \text{if } j = 2 \text{ and } i \in \{1, 4\} \\ (a_{3,3} + a_{3,4}) + (a_{3,1} + a_{3,2}) & \text{if } j = 3 \text{ and } i \in \{1, 2\} \\ 0 & \text{if } j > 4 \text{ or } i > 4 \end{cases}$$

By relation $r^{(A1)}$, this implies that generators (G2) are trivial for $g \geq 4$.

For $i = j$ and $i = j + 1$ we get

$$0 = \pm \begin{cases} (b_{1,1} + b_{1,2}) - (a_{1,1} + a_{1,2}) - (a_{1,3} + a_{1,4}) & \text{if } j = 1 \\ (b_{1,1} + b_{1,2}) + (b_{1,3} - b_{1,1}) + (a_{2,2} + a_{2,3}) + (a_{2,1} + a_{2,4}) & \text{if } j = 2 \\ (b_{1,3} - b_{1,1}) + (b_{1,1} + b_{1,4}) - (a_{3,3} + a_{3,4}) - (a_{3,1} + a_{3,2}) & \text{if } j = 3 \\ b_{1,j} + b_{1,j+1} & \text{if } j > 4. \end{cases}$$

This implies that generators (G8) for $j = 1$ generate a cyclic group, and generators (G9)–(G10) are trivial for $j = 1$.

(A4). Relation (A4) gives

$$\begin{aligned} r_i^{(A4)} : 0 &= ([b_1] + b_1[a_4] + b_1a_4[b_1] - [a_4] - a_4[b_1] - a_4b_1[a_4]) \otimes \gamma_i = \\ &= b_{1,i} + [a_4] \otimes \psi(b_1^{-1})\gamma_i + [b_1] \otimes \psi(a_4^{-1}b_1^{-1})\gamma_i - a_{4,i} \\ &\quad - [b_1] \otimes \psi(a_4^{-1})\gamma_i - [a_4] \otimes \psi(b_1^{-1}a_4^{-1})\gamma_i \\ &\equiv \begin{cases} \pm(b_{1,5} - a_{4,i}) & \text{if } i \in \{1, 2, 3\} \\ -b_{1,5} - (a_{4,4} + a_{4,5}) + (a_{4,1} + a_{4,2} + a_{4,3}) & \text{if } i = 4 \\ b_{1,5} - a_{4,1} - a_{4,2} - a_{4,3} & \text{if } i = 5 \\ b_{1,i} - a_{4,i} & \text{if } i > 5. \end{cases} \end{aligned}$$

This implies that generators (G8) for $j = 1$ are in the cyclic group generated by generators (G1).

(C6) and simplification trick. Relation (C6) gives

$$\begin{aligned} r_i^{(C6)} : 0 &= (1 + u_3b_1)([u_3] + u_3[b_1]) \otimes \gamma_i - (1 + a_1a_2a_3)([a_1] + a_1[a_2] + a_1a_2[a_3]) \otimes \gamma_i \\ &\quad - (a_1a_2a_3)^2(1 + u_1u_2u_3)([u_1] + u_1[u_2] + u_1u_2[u_3]) \otimes \gamma_i \end{aligned}$$

For $i > 4$ this easily yields

$$2b_{1,i} - 2(a_{1,i} + a_{2,i} + a_{3,i}) - 2(u_{1,i} + u_{2,i}),$$

which gives no new information.

The computations for $i \in \{1, 2, 3\}$ are a bit cumbersome, hence before we go into the details, we make a couple of observations which will greatly simplify the situation.

- Observe first that if we express $r_i^{(C6)}$, $i \in \{1, 2, 3\}$ as a combination of generators (G1)–(G5), (G8)–(G12), as we did before for other relations, we will obtain an expression without generators (G12). The reason for this is that relation (C6) does not depend on g , hence the resulting expression will be the same for all $g \geq 4$ and it will not contain $a_{j,j}$ with $j \geq 4$.
- In the obtained expression for $r_i^{(C6)}$, $i \in \{1, 2, 3\}$ we can perform the following changes, which are consequences of previously obtained relations.
 - We replace each generator (G1) with $a_{1,3}$ and each generator (G3) with $u_{1,3}$. Moreover, since $2a_{1,3} = 0$, we can assume that $a_{1,3}$ has the coefficient 0 or 1. The same for $u_{1,3}$.
 - Since generators (G2) and (G4) are trivial, we can remove all elements $a_{j,j+1}$ and $u_{j,j+1}$ (we replace them respectively by $-a_{j,j}$ and $-u_{j,j}$).
 - Since generators (G5) are trivial, we remove elements $u_{j,j}$ for $j > 1$. Observe that the obtained expression automatically does not contain $u_{1,1}$.
 - Since generators (G9)–(G10) are trivial for $j = 1$, we replace elements $b_{1,i}$ for $i > 1$ with $\pm b_{1,1}$.
- As the result of the above changes we obtain an expression for $r_i^{(C6)}$ which involves only $a_{1,3}$, $u_{1,3}$ and generators (G8), (G11). Observe also that coefficients of generators (G11) are uniquely determined by coefficients of corresponding $b_{j,1}$.

The above analysis implies that during the transformation of $r_i^{(C6)}$ we can completely ignore elements $a_{j,j}$, $a_{j,j+1}$, $u_{j,j}$, $u_{j,j+1}$, and the elements $a_{j,i}$, $u_{j,i}$ with $i \notin \{j, j+1\}$ are subject to the equivalence $\pm a_{j,i} \equiv a_{1,3}$, $\pm u_{j,i} \equiv u_{1,3}$. In particular we can ignore all elements of the form

$$[a_j] \otimes 2m, [u_j] \otimes 2m, \text{ for } m \in H_1(N_g; \mathbb{Z})$$

and as a result

$$\begin{aligned} [a_j] \otimes \psi(a_k^{\pm 1})\gamma_i &\equiv [a_j] \otimes \psi(u_k)\gamma_i, \\ [u_j] \otimes \psi(a_k^{\pm 1})\gamma_i &\equiv [u_j] \otimes \psi(u_k)\gamma_i. \end{aligned}$$

Consequently, if w is any word in letters

$$\{a_1^{\pm 1}, \dots, a_{g-1}^{\pm 1}, u_1^{\pm 1}, \dots, u_{g-1}^{\pm 1}, b_1^{\pm 1}\}$$

and w' is the word obtained from w by replacing each $a_k^{\pm 1}$ with u_k then

$$\begin{aligned} [a_j] \otimes \psi(w)\gamma_i &\equiv [a_j] \otimes \psi(w')\gamma_i, \\ [u_j] \otimes \psi(w)\gamma_i &\equiv [u_j] \otimes \psi(w')\gamma_i. \end{aligned}$$

Returning to the relation $r_i^{(C6)}$, it is equivalent to

$$\begin{aligned} r_i^{(C6)}: 0 &= ([u_3] + u_3[b_1]) \otimes (I_g + \psi(b_1^{-1}u_3))\gamma_i \\ &\quad - ([a_1] + u_1[a_2] + u_1u_2[a_3]) \otimes (I_g + \psi(u_3u_2u_1))\gamma_i \\ &\quad - ([u_1] + u_1[u_2] + u_1u_2[u_3]) \otimes (\psi((u_3u_2u_1)^2) + \psi((u_3u_2u_1)^3))\gamma_i \\ &\equiv \begin{cases} \pm 2(b_{1,1} - a_{1,1} - a_{3,3}) & \text{if } i \in \{1, 2\} \\ 0 & \text{if } i \in \{3, 4\}. \end{cases} \end{aligned}$$

Hence generator (G11) has order 2 for $j = 1$. In particular, as in the case of $[a_j]$ and $[u_j]$, we can ignore elements of the form

$$[b_1] \otimes 2m, \text{ for } m \in H_1(N_g; \mathbb{Z}).$$

Consequently,

$$[b_1] \otimes \psi(w)\gamma_i \equiv [b_1] \otimes \psi(w')\gamma_i,$$

where w' is the word obtained from w by replacing each $a_k^{\pm 1}$ with u_k .

Note that at this point we have proved Theorem 1.1 for $N = N_{4,1}$,

$$H_1(\mathcal{M}(N_{4,1}); H_1(N_{4,1}; \mathbb{Z})) \cong \langle a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(C7). Relation (C7) gives

$$\begin{aligned} r_i^{(C7)}: 0 &= ([u_5] + u_5[b_1] - [b_1] - b_1[u_5]) \otimes \gamma_i \\ &= [u_5] \otimes (I_g - \psi(b_1^{-1}))\gamma_i - [b_1] \otimes (I_g - \psi(u_5))\gamma_i \\ &= \pm \begin{cases} 0 & \text{if } i > 6 \\ b_{1,5} - b_{1,6} & \text{if } i \in \{5, 6\} \\ u_{5,1} + u_{5,2} + u_{5,3} + u_{5,4} & \text{if } i \in \{1, 2, 3, 4\}. \end{cases} \end{aligned}$$

This relation gives no new information.

(C8). Since $u_j^2 = id$ as an automorphism of $H_1(N; \mathbb{Z})$, relation (C8) gives

$$\begin{aligned} r_i^{(C8)}: 0 &= ([a_4] + a_4[u_4] + a_4u_4[a_4] + a_4u_4a_4[a_3] + a_4u_4a_4a_3[a_2]) \otimes \gamma_i \\ &\quad + (a_4u_4a_4a_3a_2[a_1] + a_4u_4a_4a_3a_2a_1[u_1] + a_4u_4a_4a_3a_2a_1u_1[u_2]) \otimes \gamma_i \\ &\quad + (a_4u_4a_4a_3a_2a_1u_1u_2[u_3] + a_4u_4a_4a_3a_2a_1u_1u_2u_3[u_4]) \otimes \gamma_i \\ &\quad + (a_4u_4a_4a_3a_2a_1u_1u_2u_3u_4[b_1] - [b_1] - b_1[a_4] - b_1a_4[u_4]) \otimes \gamma_i \\ &= ([a_4] + u_4[u_4] + [a_4] + u_4[a_3] + u_4u_3[a_2] + u_4u_3u_2[a_1]) \otimes \gamma_i \\ &\quad + (u_4u_3u_2u_1[u_1] + u_4u_3u_2[u_2] + u_4u_3[u_3] + u_4[u_4]) \otimes \gamma_i \\ &\quad + ([b_1] - [b_1] - b_1[a_4] - b_1u_4[u_4]) \otimes \gamma_i \\ &= (u_4[a_3] + u_4u_3[a_2] + u_4u_3u_2[a_1] + u_4u_3u_2u_1[u_1]) \otimes \gamma_i \\ &\quad + (u_4u_3u_2[u_2] + u_4u_3[u_3] - b_1[a_4] - b_1u_4[u_4]) \otimes \gamma_i. \end{aligned}$$

Now it is straightforward to check that this relation gives no new information.

(A5). Relation (A5) gives

$$\begin{aligned}
r_i^{(A5)}: 0 &= \left(\sum_{p=0}^9 (a_2 a_3 a_4 b_1)^p \right) ([a_2] + a_2[a_3] + a_2 a_3[a_4] + a_2 a_3 a_4[b_1]) \otimes \gamma_i \\
&\quad - \left(\sum_{p=0}^5 (a_1 a_2 a_3 a_4 b_1)^p \right) ([a_1] + a_1[a_2] + a_1 a_2[a_3] + a_1 a_2 a_3[a_4] \\
&\quad \quad + a_1 a_2 a_3 a_4[b_1]) \otimes \gamma_i \\
&= ([a_2] + a_2[a_3] + a_2 a_3[a_4] + a_2 a_3 a_4[b_1]) \otimes \left(\sum_{p=0}^9 A^p \right) \gamma_i \\
&\quad - ([a_1] + a_1[a_2] + a_1 a_2[a_3] + a_1 a_2 a_3[a_4] + a_1 a_2 a_3 a_4[b_1]) \otimes \left(\sum_{p=0}^5 B^p \right),
\end{aligned}$$

where $A = \psi(b_1^{-1} a_4^{-1} a_3^{-1} a_2^{-1})$ and $B = A \circ \psi(a_1^{-1})$.

It is straightforward to verify that

$$A = \begin{bmatrix} 2 & -2 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \oplus I_{g-5}, \quad B = \begin{bmatrix} 2 & -2 & 1 & -1 & 1 \\ 2 & -1 & 0 & -1 & 1 \\ 2 & -1 & 1 & -2 & 1 \\ 2 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus I_{g-5}$$

and

$$\sum_{p=0}^9 A^p = 10C, \quad \sum_{p=0}^5 B^p = 6C,$$

where

$$C = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \oplus I_{g-5}.$$

Hence this relation gives no new information.

Note that at this point we proved Theorem 1.1 for $N = N_{5,1}$,

$$H_1(\mathcal{M}(N_{5,1}); H_1(N_{5,1}; \mathbb{Z})) = \langle a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(A6). Relation (A6) gives

$$\begin{aligned}
r_i^{(A6)} : 0 &= \left(\sum_{p=0}^{11} (a_2 a_3 a_4 a_5 a_6 b_1)^p \right) ([a_2] + a_2[a_3] + a_2 a_3[a_4] + a_2 a_3 a_4[a_5] \\
&\quad + a_2 a_3 a_4 a_5[a_6] + a_2 a_3 a_4 a_5 a_6[b_1]) \otimes \gamma_i \\
&\quad - \left(\sum_{p=0}^8 (a_1 a_2 a_3 a_4 a_5 a_6 b_1)^p \right) ([a_1] + a_1[a_2] + a_1 a_2[a_3] + a_1 a_2 a_3[a_4] \\
&\quad + a_1 a_2 a_3 a_4[a_5] + a_1 a_2 a_3 a_4 a_5[a_6] + a_1 a_2 a_3 a_4 a_5 a_6[b_1]) \otimes \gamma_i \\
&= ([a_2] + a_2[a_3] + \cdots + a_2 a_3 a_4 a_5 a_6[b_1]) \otimes \left(\sum_{p=0}^{11} A^p \right) \gamma_i \\
&\quad - ([a_1] + a_1[a_2] + \cdots + a_1 a_2 a_3 a_4 a_5 a_6[b_1]) \otimes \left(\sum_{p=0}^8 B^p \right),
\end{aligned}$$

where $A = \psi(b_1^{-1} a_6^{-1} a_5^{-1} a_4^{-1} a_3^{-1} a_2^{-1})$ and $B = A \circ \psi(a_1^{-1})$.

It is straightforward to verify that

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus I_{g-7}, \\
B &= \begin{bmatrix} 2 & -2 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & -2 & 1 & 0 & 0 \\ 2 & -1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus I_{g-7}
\end{aligned}$$

and

$$\sum_{p=0}^{11} A^p = 12C, \quad \sum_{p=0}^8 B^p = 9C,$$

where

$$C = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ \dots & & & & & & \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}_{7 \times 7} \oplus I_{g-7}.$$

This implies that for $i \leq 7$ relation $r_i^{(A6)}$ is equivalent to

$$\begin{aligned} r_i^{(A6)}: 0 &= ([a_1] + u_1[a_2] + u_1u_2[a_3] + u_1u_2u_3[a_4] + u_1u_2u_3u_4[a_5] \\ &\quad + u_1u_2u_3u_4u_5[a_6] + u_1u_2u_3u_4u_5u_6[b_1]) \otimes (\gamma_1 + \gamma_2 + \cdots + \gamma_7) \\ &= ([a_1] + [a_2] + [a_3] + [a_4] + [a_5] + [a_6] + [b_1]) \otimes (\gamma_1 + \gamma_2 + \cdots + \gamma_7) \\ &\equiv 6 \cdot 5a_{1,3} + b_{1,5} + b_{1,6} + b_{1,7} \equiv a_{1,3}. \end{aligned}$$

Hence generator (G1) is trivial for $g \geq 7$. For $i > 7$ we get exactly the same conclusion as above.

Note that at this point we proved Theorem 1.1 for $N = N_{g,1}$, where $g \geq 7$ is odd,

$$H_1(\mathcal{M}(N_{g,1}); H_1(N_{g,1}; \mathbb{Z})) = \langle u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for } g \geq 7 \text{ odd.}$$

(A8) for $j = 1$. If $j = 1$, relation (A8) gives

$$\begin{aligned} r_{1:i}^{(A8)}: 0 &= b_{2,i} + ([a_1] + a_1[a_2] + a_1a_2[a_3] + a_1a_2a_3[a_4] + a_1a_2a_3a_4[a_5]) \\ &\quad \otimes (I_g + M + M^2 + \cdots + M^5)b_2^{-1}\gamma_i \\ &\quad - ([a_1] + a_1[a_2] + a_1a_2[a_3] + a_1a_2a_3[a_4] + a_1a_2a_3a_4[a_5] + a_1a_2a_3a_4a_5[b_1]) \\ &\quad \otimes (I_g + N + N^2 + \cdots + N^4)\gamma_i. \end{aligned}$$

Where $M = \psi(a_5^{-1}a_4^{-1}a_3^{-1}a_2^{-1}a_1^{-1})$ and $N = \psi(b_1^{-1})M$. Observe that we can reduce all coefficients in the above relation modulo 2, hence as matrices for M and N we can take

$$\begin{aligned} M = \psi(u_5u_4u_3u_2u_1) \pmod{2} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus I_{g-6}, \\ N = \psi(b_1^{-1})M \pmod{2} &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus I_{g-6}. \end{aligned}$$

We transform $r_{1:i}^{(A8)}$ further as follows.

$$\begin{aligned} r_{1:i}^{(A8)}: 0 &\equiv b_{2,i} + ([a_1] \otimes (A - B) + [a_2] \otimes \psi(u_1)(A - B) + [a_3] \otimes \psi(u_2u_1)(A - B) \\ &\quad + [a_4] \otimes \psi(u_3u_2u_1)(A - B) + [a_5] \otimes \psi(u_4u_3u_2u_1)(A - B))\gamma_i \\ &\quad - [b_1] \otimes \psi(u_5u_4u_3u_2u_1)B\gamma_i, \end{aligned}$$

where

$$B = \sum_{p=0}^4 N^p \pmod{2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \oplus I_{g-6},$$

$$A - B = \sum_{p=0}^5 M^p - B \pmod{2} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \oplus I_{g-6}.$$

If $i > 6$ this implies that generators (G8) are trivial for $j = 2$, and if $i \leq 6$, we have

$$r_{1:i}^{(A8)} : 0 \equiv \begin{cases} b_{2,i} - b_{1,2} - b_{1,4} - b_{1,6} & \text{if } i \in \{1, 3, 5\} \\ b_{2,i} - b_{1,1} - b_{1,3} - b_{1,5} & \text{if } i \in \{2, 4, 6\} \end{cases}$$

$$\equiv \begin{cases} (b_{2,1} - a_{1,1} - a_{3,3} - a_{5,5}) - b_{1,6} & \text{if } i = 1 \\ (b_{2,i} - b_{2,1}) + (b_{2,1} - a_{1,1} - a_{3,3} - a_{5,5}) - b_{1,6} & \text{if } i \in \{3, 5\} \\ (b_{2,i} + b_{2,1}) - (b_{2,1} - a_{1,1} - a_{3,3} - a_{5,5}) - b_{1,5} & \text{if } i \in \{2, 4, 6\}. \end{cases}$$

Hence if $j = 1$ then generators (G9)–(G10) are trivial, and generator (G8) is in the cyclic group of order 2 generated by generators (G1).

(A9a). Relation (A9a) gives

$$r_i^{(A9a)} : 0 = [b_2] \otimes (I_g - \psi(b_1^{-1})) \gamma_i + [b_1] \otimes (\psi(b_2^{-1}) - I_g) \gamma_i$$

$$\equiv \begin{cases} -(b_{2,4} + b_{2,1}) - (b_{2,3} - b_{2,1}) - (b_{2,2} + b_{2,1}) & \text{if } i \leq 4 \\ 0 & \text{if } i > 4. \end{cases}$$

This relation gives no new information.

Note that at this point we proved Theorem 1.1 for $N = N_{6,1}$,

$$H_1(\mathcal{M}(N_{6,1}); H_1(N_{6,1}; \mathbb{Z})) \cong \langle a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(A8) for $j > 1$. We transform relation (A8) in exactly the same way, as in the case of $j = 1$, however now we know that generators (G1) are trivial (since $g \geq 8$), hence we can completely ignore elements $a_{j,i}$. Moreover, we can inductively assume that generators (G8)–(G10) containing $b_{k,i}$ with $k < j + 1$ are trivial, and generators (G11) containing $b_{k,1}$ with $k < j + 1$ have order at most 2. Hence $r_{j:i}^{(A8)}$ takes form

$$r_{j:i}^{(A8)} : 0 \equiv b_{j+1,i} + [b_{j-1}] \otimes (A - B) \gamma_i - [b_j] \otimes \psi(u_{2j+3} u_{2j+2} u_{2j+1} u_{2j} b_{j-1}^{-1}) B \gamma_i,$$

where $M = \psi(u_{2j+3}u_{2j+2}u_{2j+1}u_{2j}b_{j-1}^{-1})$, $N = \psi(b_j^{-1})M$ and

$$A = \sum_{p=0}^5 M^p, \quad B = \sum_{p=0}^4 N^p.$$

Now we will identify matrices for M, N, A and B – the computations are straightforward, but a bit tedious. Using formulas for u_j and b_j , we get

$$\begin{aligned} M\gamma_i &\equiv \begin{cases} \gamma_i + (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+4} & \text{if } i < 2j \\ \gamma_1 + \cdots + \gamma_{2j-1} & \text{if } i = 2j \\ \gamma_{i-1} & \text{if } 2j+1 \leq i \leq 2j+4 \\ \gamma_i & \text{if } i > 2j+4 \end{cases} \\ M^2\gamma_i &\equiv \begin{cases} \gamma_i + (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+3} & \text{if } i < 2j \\ \gamma_{2j+4} & \text{if } i = 2j \\ \gamma_1 + \cdots + \gamma_{2j-1} & \text{if } i = 2j+1 \\ \gamma_{i-2} & \text{if } 2j+2 \leq i \leq 2j+4 \\ \gamma_i & \text{if } i > 2j+4 \end{cases} \\ M^4\gamma_i &\equiv \begin{cases} \gamma_i + (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+1} & \text{if } i < 2j \\ \gamma_{i+2} & \text{if } 2j \leq i \leq 2j+2 \\ \gamma_1 + \cdots + \gamma_{2j-1} & \text{if } i = 2j+3 \\ \gamma_{2j} & \text{if } i = 2j+4 \\ \gamma_i & \text{if } i > 2j+4 \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} (I_g + M)\gamma_i &\equiv \begin{cases} (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+4} & \text{if } i < 2j \\ \gamma_1 + \cdots + \gamma_{2j-1} + \gamma_{2j} & \text{if } i = 2j \\ \gamma_{i-1} + \gamma_i & \text{if } 2j+1 \leq i \leq 2j+4 \\ 0 & \text{if } i > 2j+4 \end{cases} \\ (I_g + M^2 + M^4)\gamma_i &\equiv \begin{cases} \gamma_i + \gamma_{2j+1} + \gamma_{2j+3} & \text{if } i < 2j \\ \gamma_{2j} + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i \in \{2j, 2j+2, 2j+4\} \\ (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+1} + \gamma_{2j+3} & \text{if } i \in \{2j+1, 2j+3\} \\ \gamma_i & \text{if } i > 2j+4 \end{cases} \\ A = (I_g + M)(I_g + M^2 + M^4)\gamma_i &\equiv \begin{cases} \gamma_1 + \cdots + \gamma_{2j+4} & \text{if } i \leq 2j+4 \\ 0 & \text{if } i > 2j+4 \end{cases} \end{aligned}$$

Obtained formula and our inductive assumption about generators (G8)–(G10) implies that we can ignore A in the relation $r_{j:i}^{(A8)}$, hence it takes form

$$r_{j:i}^{(A8)} : 0 \equiv b_{j+1,i} + [b_{j-1}] \otimes B\gamma_i - [b_j] \otimes \psi(u_{2j+3}u_{2j+2}u_{2j+1}u_{2j}b_{j-1}^{-1})B\gamma_i.$$

Now we concentrate on the formulas for $N = \psi(b_j^{-1})M$ and B .

$$N\gamma_i \equiv \begin{cases} \gamma_i + (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+4} & \text{if } i < 2j \\ \gamma_{2j} + \gamma_{2j+1} + \gamma_{2j+2} & \text{if } i = 2j \\ \gamma_{i-1} + (\gamma_1 + \cdots + \gamma_{2j+2}) & \text{if } 2j+1 \leq i \leq 2j+3 \\ \gamma_{2j+3} & \text{if } i = 2j+4 \\ \gamma_i & \text{if } i > 2j+4 \end{cases}$$

$$N^2\gamma_i \equiv \begin{cases} \gamma_i + (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+3} & \text{if } i < 2j \\ \gamma_{2j+2} & \text{if } i = 2j \\ \gamma_{2j} + \gamma_{2j+1} + \gamma_{2j+4} & \text{if } i = 2j+1 \\ \gamma_{i-2} + (\gamma_1 + \cdots + \gamma_{2j+1}) + \gamma_{2j+4} & \text{if } i \in \{2j+2, 2j+3\} \\ (\gamma_1 + \cdots + \gamma_{2j+1}) & \text{if } i = 2j+4 \\ \gamma_i & \text{if } i > 2j+4 \end{cases}$$

Hence

$$(N + N^2)\gamma_i \equiv \begin{cases} \gamma_{2j+3} + \gamma_{2j+4} & \text{if } i < 2j \\ \gamma_{2j} + \gamma_{2j+1} & \text{if } i = 2j \\ (\gamma_1 + \cdots + \gamma_{2j}) + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i = 2j+1 \\ \gamma_{2j} + \gamma_{2j+1} + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i = 2j+2 \\ \gamma_{2j+1} + \gamma_{2j+4} & \text{if } i = 2j+3 \\ (\gamma_1 + \cdots + \gamma_{2j+1}) + \gamma_{2j+3} & \text{if } i = 2j+4 \\ 0 & \text{if } i > 2j+4 \end{cases}$$

$$(N + N^2)N^2\gamma_i \equiv \begin{cases} \gamma_{2j+1} + \gamma_{2j+4} & \text{if } i < 2j \\ \gamma_{2j} + \gamma_{2j+1} + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i = 2j \\ \gamma_{2j} + \gamma_{2j+2} + \gamma_{2j+3} + \gamma_{2j+4} & \text{if } i = 2j+1 \\ \gamma_{2j+1} + \gamma_{2j+2} & \text{if } i = 2j+2 \\ (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+4} & \text{if } i = 2j+3 \\ (\gamma_1 + \cdots + \gamma_{2j+3}) + \gamma_{2j} & \text{if } i = 2j+4 \\ 0 & \text{if } i > 2j+4. \end{cases}$$

Therefore $B = I_g + (N + N^2) + (N + N^2)N^2$ is equal to

$$B\gamma_i \equiv \begin{cases} \gamma_i + \gamma_{2j+1} + \gamma_{2j+3} & \text{if } i < 2j \\ \gamma_{2j} + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i \in \{2j, 2j+2, 2j+4\} \\ (\gamma_1 + \cdots + \gamma_{2j-1}) + \gamma_{2j+1} + \gamma_{2j+3} & \text{if } i \in \{2j+1, 2j+3\} \\ \gamma_i & \text{if } i > 2j+4, \end{cases}$$

and $B' = \psi(u_{2j+3}u_{2j+2}u_{2j+1}u_{2j}b_{j-1}^{-1})B$ is equal to

$$B'\gamma_i \equiv \begin{cases} \gamma_i + (\gamma_1 + \dots + \gamma_{2j}) + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i < 2j \\ (\gamma_1 + \dots + \gamma_{2j-1}) + \gamma_{2j+1} + \gamma_{2j+3} & \text{if } i \in \{2j, 2j+2, 2j+4\} \\ \gamma_{2j} + \gamma_{2j+2} + \gamma_{2j+4} & \text{if } i \in \{2j+1, 2j+3\} \\ \gamma_i & \text{if } i > 2j+4. \end{cases}$$

If $i > 2j+4$ this implies that generators (G8) are trivial and if $i \leq 2j+4$ we get

$$\begin{aligned} r_{j:i}^{(A8)} : 0 &\equiv \begin{cases} b_{j+1,i} + b_{j-1,i} + b_{j-1,2j+1} + b_{j-1,2j+3} \\ \quad - (b_{j,i} + b_{j,2j+2}) - b_{j,2j+4} & \text{if } i < 2j \\ b_{j+1,i} + b_{j-1,2j} + b_{j-1,2j+2} + b_{j-1,2j+4} \\ \quad - (b_{j,2j-1} + b_{j,2j+1}) - b_{j,2j+3} & \text{if } i \in \{2j, 2j+2, 2j+4\} \\ b_{j+1,i} + b_{j-1,2j-1} + b_{j-1,2j+1} + b_{j-1,2j+3} \\ \quad - (b_{j,2j} + b_{j,2j+2}) - b_{j,2j+4} & \text{if } i \in \{2j+1, 2j+3\} \end{cases} \\ &\equiv \begin{cases} b_{j+1,i} + b_{j-1,i} & \text{if } i < 2j \\ b_{j+1,i} + b_{j-1,2j} & \text{if } i \in \{2j, 2j+2, 2j+4\} \\ b_{j+1,i} + b_{j-1,2j-1} & \text{if } i \in \{2j+1, 2j+3\} \end{cases} \\ &\equiv \begin{cases} (b_{j+1,1} - a_{1,1} - \dots - a_{2j+1,2j+1}) \\ \quad + (b_{j-1,1} - a_{1,1} - \dots - a_{2j-3,2j-3}) & \text{if } i = 1 \\ (b_{j+1,i} \pm b_{j+1,1}) \pm (b_{j+1,1} - a_{1,1} - \dots - a_{2j+1,2j+1}) \\ \quad \pm (b_{j-1,1} - a_{1,1} - \dots - a_{2j-3,2j-3}) \pm (b_{j-1,2} + b_{j-1,1}) & \text{if } 1 < i \leq 2j+4. \end{cases} \end{aligned}$$

Hence we inductively proved that generators (G8)–(G10) are trivial, and generators (G11) are either trivial or they are in the cyclic group of order 2 generated by $b_{1,1} - a_{1,1} - a_{3,3}$.

(A9b). Relation (A9b) gives

$$\begin{aligned} r_i^{(A9b)} : 0 &= [b_{\frac{g-2}{2}}] \otimes (I_g - \psi(a_{g-5}^{-1})) \gamma_i + [a_{g-5}] \otimes \left(\psi \left(b_{\frac{g-2}{2}}^{-1} \right) - I_g \right) \gamma_i \\ &\equiv [b_{\frac{g-2}{2}}] \otimes (I_g - \psi(u_{g-5})) \gamma_i \\ &\equiv \begin{cases} (b_{\frac{g-2}{2},g-4} + b_{\frac{g-2}{2},1}) - (b_{\frac{g-2}{2},g-5} - b_{\frac{g-2}{2},1}) & \text{if } i \in \{g-5, g-4\} \\ 0 & \text{if } i \notin \{g-5, g-4\}. \end{cases} \end{aligned}$$

This relation gives no new information.

Note that at this point we proved that

$$H_1(\mathcal{M}(N_{g,1}); H_1(N_{g,1}; \mathbb{Z})) = \langle u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for } g \geq 8 \text{ even,}$$

which completes the proof of Theorem 1.1 for $N = N_{g,1}$.

Now we turn to the case of a closed surface $N = N_g$ where $g \geq 4$.

(B3). Relation (B3) has been carefully studied in Section 5.4 of [15] (it is called (E3) there). If g is even, it gives

$$r_i^{(B3)}: 0 = 2a_{1,1} + 2a_{3,3} + \cdots + 2a_{g-1,g-1},$$

which means that generator (G12) is trivial, and if g is odd, it gives

$$\begin{aligned} r_i^{(B3)}: 0 &= 2a_{1,1} + 2a_{3,3} + \cdots + 2a_{g-2,g-2} - \varrho_g \\ &= -(2a_{g-1,g-1} + \varrho_{g-1} + \varrho_g) + (2a_{g-2,g-2} + \varrho_{g-2} + \varrho_{g-1}) \\ &\quad \cdots + (2a_{1,1} + \varrho_1 + \varrho_2) + (2a_{2,2} + 2a_{4,4} + \cdots + 2a_{g-1,g-1}) - \varrho_1 \\ &\equiv (a_{1,1} + 2a_{2,2} + 2a_{4,4} + \cdots + 2a_{g-1,g-1} - u_{1,1}) - (a_{1,1} + \varrho_1 - u_{1,1}), \end{aligned}$$

which means that generator (G12) is superfluous.

(D1). Relation (D1) gives

$$\begin{aligned} r_{j,i}^{(D1)}: 0 &= ([\varrho] + \varrho[a_j] - [a_j] - a_j[\varrho]) \otimes \gamma_i \\ &= \begin{cases} -2a_{j,j} - \varrho_j - \varrho_{j+1} & \text{if } i = j \\ (2a_{j,j} + \varrho_j + \varrho_{j+1}) - 2(a_{j,j} + a_{j,j+1}) & \text{if } i = j + 1 \\ -2a_{j,i} & \text{if } i \notin \{j, j + 1\}. \end{cases} \end{aligned}$$

Hence generators (G6) are trivial.

(D2). Relation (D2) gives

$$\begin{aligned} r_{j,i}^{(D2)}: 0 &= ([u_1] + u_1[\varrho] + u_1\varrho[u_1] - [\varrho]) \otimes \gamma_i \\ &= \pm \begin{cases} (2a_{1,1} + \varrho_1 + \varrho_2) - (u_{1,1} + u_{1,2}) & \text{if } i \in \{1, 2\} \\ -2(a_{1,1} + \varrho_1 - u_{1,1}) & \text{if } i > 2. \\ 0 & \end{cases} \end{aligned}$$

Hence generator (G7) has order 2.

(E). Relation (E) gives no new information.

$$r_{j,i}^{(E)}: 0 = ([\varrho] + \varrho[\varrho]) \otimes \gamma_i = \varrho_i - \varrho_i = 0.$$

(F). Before we deal with relation (F) let us try to establish some simplification rules, as in the case of relations not depending on g . After rewriting relation (F) we will obtain a combination of generators (G1)–(G3) ((G3) only with $j = 1$), (G6)–(G7) and (G12). Since generators (G2) and (G6) are trivial, we can assume that in the obtained relation there are no elements $a_{j,j+1}$ for $j = 1, \dots, g-1$ nor ϱ_j for $j \geq 2$. Using generator (G12) we can also remove all elements $a_{g-1,g-1}$, and since generators (G1) and (G3) generate cyclic groups, we can assume that in the obtained relation we have $a_{j,i}$ and $u_{j,i}$ only if $j = 1$ and $i \in \{1, 3\}$. As a result we will obtain a combination of $a_{1,3}, u_{1,3}$ and generator (G7). All this elements have order 2, hence we can compute coefficients modulo 2. Moreover, the coefficient of generator (G7) is completely determined by coefficient of ϱ_1 , so we are not really interested in coefficients of $a_{1,1}$ and $u_{1,1}$.

For $n \in \{1, \dots, g\}$ we define

$$X_n = \psi(u_{n-1} \cdots u_2 u_1 \cdot u_1) = \begin{cases} \psi(u_1) & \text{if } n = 1 \\ I_g & \text{if } n = 2 \\ \psi(u_{n-1} \cdots u_2) & \text{if } n \in \{3, \dots, g-1\}. \end{cases}$$

Relation (F) gives

$$\begin{aligned} r_{j:i}^{(F)} : 0 &= \left(\sum_{p=0}^{g-2} (u_1 a_1 a_2 a_3 \cdots a_{g-1} \varrho)^p \right) [u_1] \otimes \gamma_i \\ &\quad + \sum_{n=1}^{g-1} \left(\sum_{p=0}^{g-2} (u_1 a_1 a_2 a_3 \cdots a_{g-1} \varrho)^p \right) (u_1 a_1 \cdots a_{n-1}) [a_n] \otimes \gamma_i \\ &\quad + \left(\sum_{p=1}^{g-1} (u_1 a_1 a_2 a_3 \cdots a_{g-1} \varrho)^p \right) [\varrho] \otimes \gamma_i \\ &\equiv [u_1] \otimes \sum_{p=0}^{g-2} X_g^p \gamma_i + \sum_{n=1}^{g-1} [a_n] \otimes X_n \sum_{p=0}^{g-2} X_g^p \gamma_i + [\varrho] \otimes \sum_{p=1}^{g-1} X_g^p \gamma_i. \end{aligned}$$

Observe now that if $n \geq 3$ then X_n acts on a basis $(\gamma_1, \dots, \gamma_g)$ as the cycle

$$\gamma_n \mapsto \gamma_{n-1} \mapsto \dots \mapsto \gamma_3 \mapsto \gamma_2 \mapsto \gamma_n$$

of order $n-1$. In particular X_g is the cycle

$$\gamma_g \mapsto \gamma_{g-1} \mapsto \dots \mapsto \gamma_3 \mapsto \gamma_2 \mapsto \gamma_g$$

of order $g-1$, hence

$$\sum_{p=0}^{g-2} X_g^p \gamma_i = \sum_{p=1}^{g-1} X_g^p \gamma_i = \begin{cases} (g-1)\gamma_1 & \text{if } i = 1 \\ \gamma_2 + \dots + \gamma_g & \text{if } i \in \{2, \dots, g\}. \end{cases}$$

Therefore for $i = 1$ we have

$$\begin{aligned} r_{j:1}^{(F)} : 0 &= (g-1) \left(u_{1,1} + \sum_{n=1}^{g-1} [a_n] \otimes X_n \gamma_1 + \varrho_1 \right) \\ &= (g-1)(u_{1,1} + a_{1,1} + a_{2,1} + \dots + a_{g-1,1} + \varrho_1) \\ &\equiv \begin{cases} 0 & \text{if } g \text{ odd} \\ (a_{1,1} + \varrho_1 - u_{1,1}) & \text{if } g \text{ even.} \end{cases} \end{aligned}$$

For $i > 1$ we have

$$\begin{aligned}
 r_{j:i}^{(F)}: 0 &= u_{1,2} + \cdots u_{1,g} + (a_{1,1} + a_{1,3} + \cdots + a_{1,g}) + \\
 &\quad + \sum_{n=2}^{g-1} (a_{n,2} + \cdots + a_{n,g}) + \varrho_2 + \cdots + \varrho_g \\
 &\equiv \begin{cases} u_{1,3} + a_{1,3} + (a_{1,1} + 2a_{2,2} + \cdots + 2a_{g-1,g-1} - u_{1,1}) & \text{if } g \text{ is odd} \\ (a_{1,1} + \varrho_1 - u_{1,1}) & \text{if } g \text{ is even.} \end{cases}
 \end{aligned}$$

Hence in all cases generators (G7) and (G12) are superfluous.

This proves that

$$\begin{aligned}
 H_1(\mathcal{M}(N_g); H_1(N_g; \mathbb{Z})) &= \begin{cases} \langle a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle & \text{if } g \in \{4, 5, 6\} \\ \langle u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3} \rangle & \text{if } g \geq 7 \end{cases} \\
 &= \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \in \{4, 5, 6\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \geq 7. \end{cases}
 \end{aligned}$$

which completes the proof of Theorem 1.1.

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